Algorithmic needs for Fusion Magnetohydrodynamics (MHD) and other Predominantly Hyperbolic Systems of Equations

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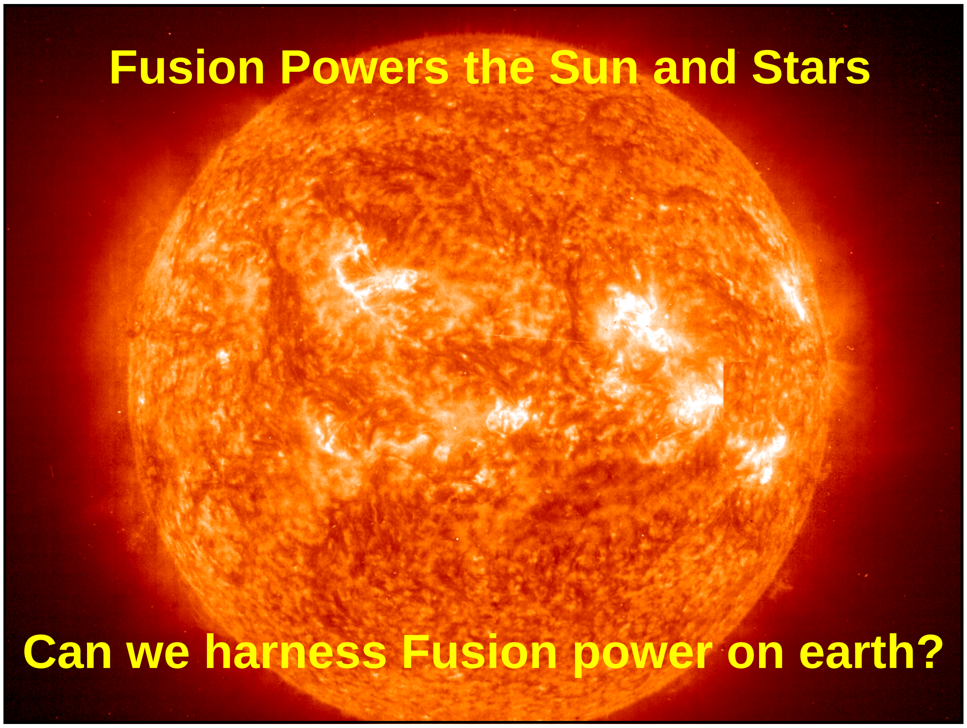
Outline

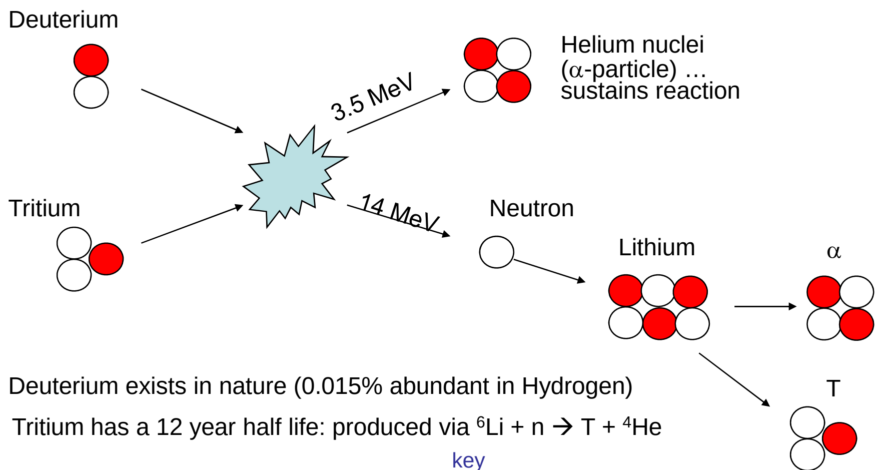
- Fusion…the ultimate energy source
- Basics of Magnetic Fusion
- The Role of Computer Simulation
- Implicit Solution of Hyperbolic Equations
- Approach of the CEMM codes
- A proposed 3D Solution Technique
- Other efforts within CEMM



The Case for Fusion Energy

- Worldwide demand for energy continues to increase
  - Due to population increases and economic development
  - Most population growth and energy demand is in urban areas
    - Implies need for large, centralized power generation
- Worldwide oil and gas production is near or past peak
  - Need for alternative source: coal, fission, fusion
- Increasing evidence that release of greenhouse gases is causing global climate change . . . “Global warming”
  - Historical data and 100+ year detailed climate projections
  - This makes nuclear (fission or fusion) preferable to fossil (coal)
- Fusion has some advantages over fission that could become critical:
  - Inherent safety (no China syndrome)
  - No weapons proliferation considerations (security)
  - Greatly reduced waste disposal problems (no Yucca Mt.)

Controlled Fusion uses isotopes of Hydrogen in a High Temperature Ionized Gas (Plasma)



Deuterium exists in nature (0.015% abundant in Hydrogen)

Tritium has a 12 year half life: produced via $^6Li + n \rightarrow T + {}^4He$

Lithium is naturally abundant

key
- proton
- neutron

Controlled Fusion Basics

Create a mixture of D and T (plasma), heat it to high temperature, and the D and T will fuse to produce energy.

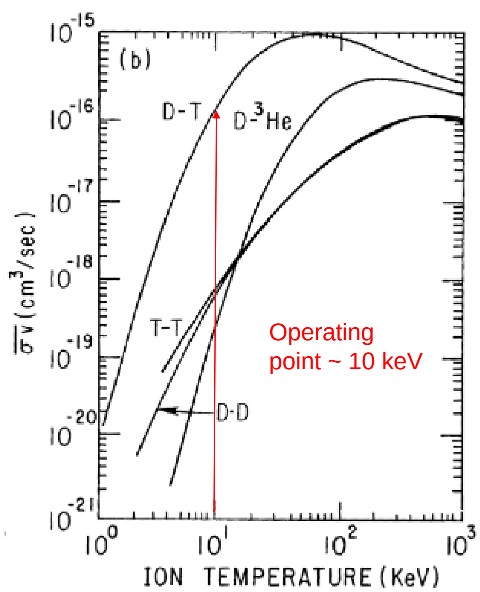
$P_{DT} = n_D n_T <\sigma v>(U_\alpha + U_n)$

at 10 keV, $<\sigma v> \sim T^2$

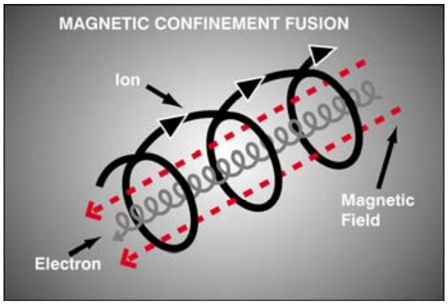
$P_{DT} \sim$ (plasma pressure)^2

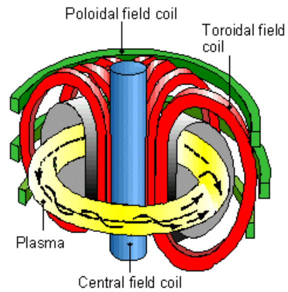
Need ~ 5 atmosphere @ 10 keV

Note: 1 keV = 10,000,000 deg(K)



Toroidal Magnetic Confinement



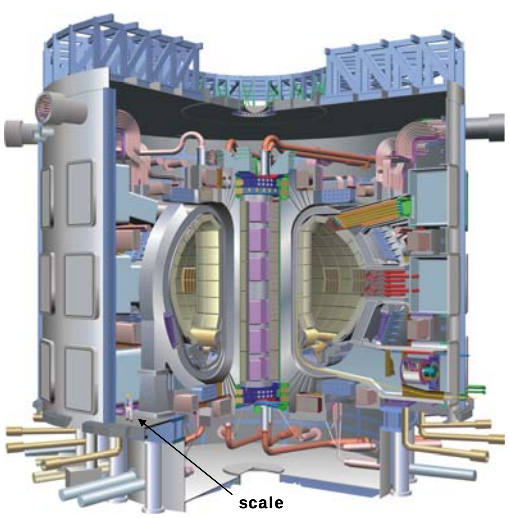


Charged particles have helical orbits in a magnetic field; they describe circular orbits perpendicular to the field and free-stream in the direction of the field.

TOKAMAK creates toroidal magnetic fields to confine particles in the 3rd dimension. Includes an induced toroidal plasma current to heat and confine the plasma

"TOKAMAK": Russian abbreviation for "toroidal chamber"

The U.S. is an official partner in ITER

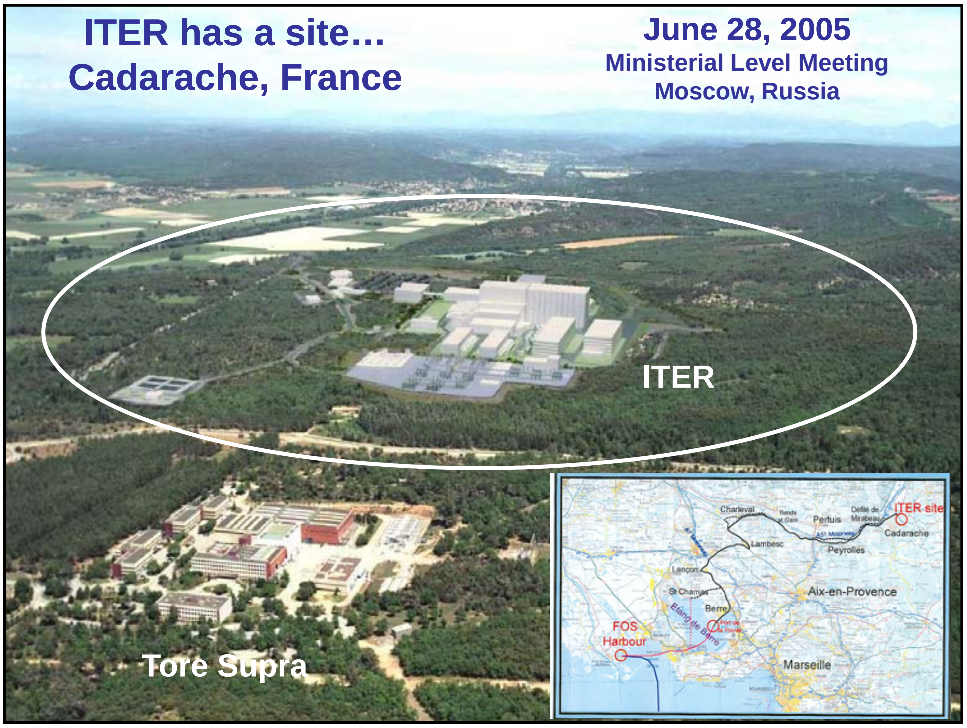


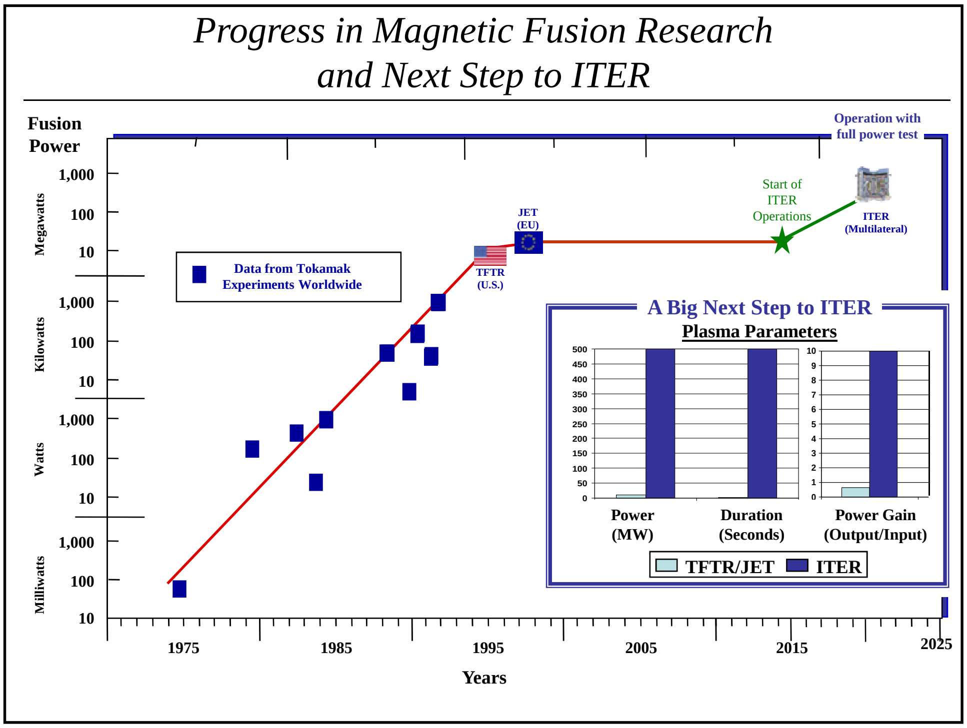
- 500 MW fusion output
- Cost: $ 5-10 B
- To begin operation in 2015

International
Thermonuclear
Experimental **R**eactor:

- European Union
- Japan
- United States
- Russia
- Korea
- China
- India

- World's largest tokamak
- all super-conducting coils

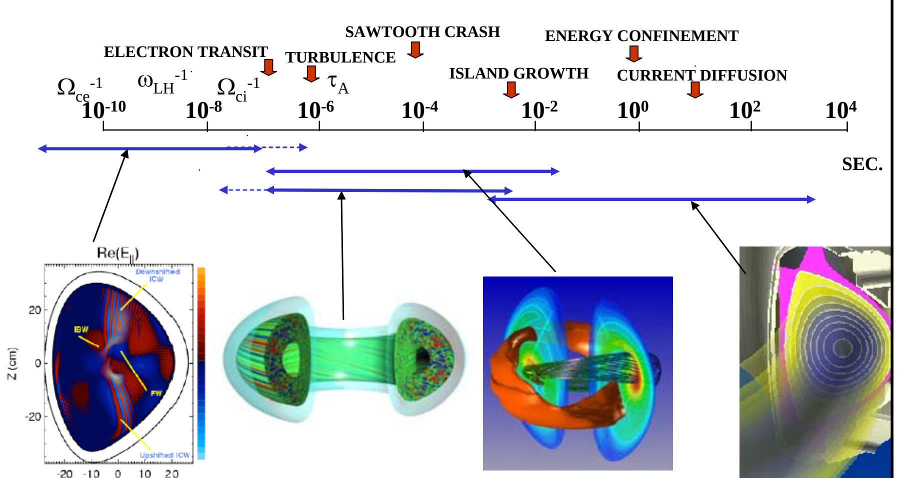

ITER has a site…
Cadarache, France
June 28, 2005
Ministerial Level Meeting
Moscow, Russia
ITER
Tore Supra
ITER site
Cadarache
Aix-en-Provence
FOS
Harbour
Marseille


Progress in Magnetic Fusion Research
and Next Step to ITER
Fusion Power
Megawatts
Kilowatts
Watts
Milliwatts
Data from Tokamak Experiments Worldwide
TFTR (U.S.)
JET (EU)
Start of ITER Operations
Operation with full power test
ITER (Multilateral)
A Big Next Step to ITER
Plasma Parameters
Power (MW)
Duration (Seconds)
Power Gain (Output/Input)
TFTR/JET
ITER
1975
1985
1995
2005
2015
2025
Years

Simulations are needed in 4 areas

- How to heat the plasma to thermonuclear temperatures (~ 100,000,000^{o} C)
- How to reduce the background turbulence
- How to eliminate device-scale instabilities
- How to optimize the operation of the whole device

These 4 areas address different timescales and are normally studied using different codes



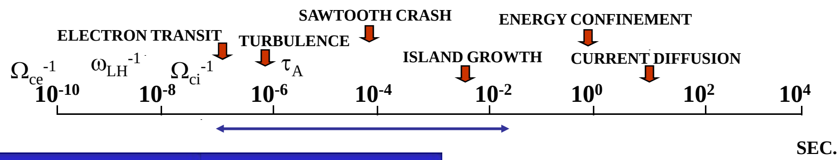
(a) RF codes

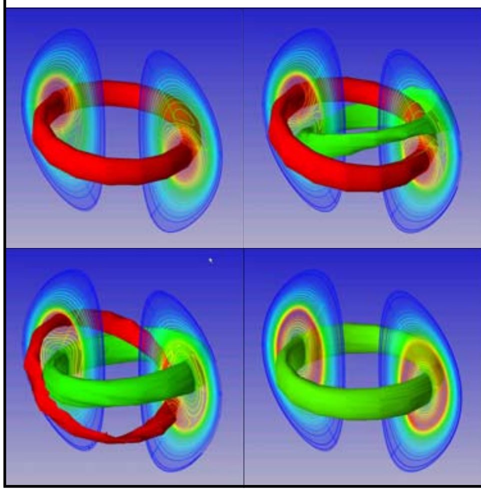
(b) Micro-turbulence codes

(c) Extended-MHD codes

(d) Transport C

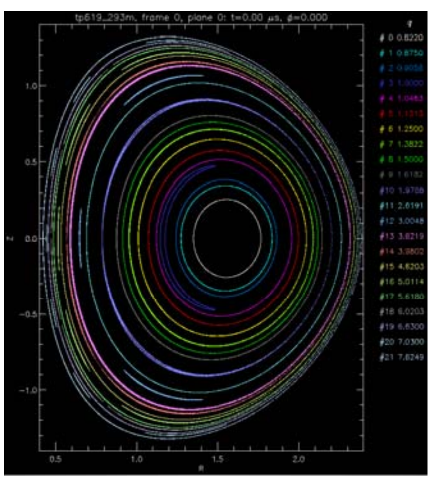
Extended MHD Codes solve 3D fluid equations for device-scale stability





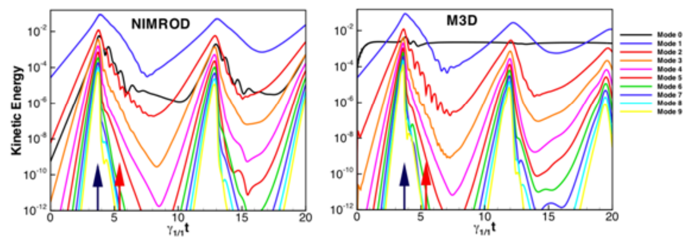
- Sawtooth cycle is one example of global phenomena that need to be understood
- Can cause degradation of confinement, or plasma termination
- The X-MHD codes typically exhibit good parallel scaling to 500-1000 processors
- Running time is dominated by elliptic solves
- Need to run for many time steps

Show Quicktime Movie

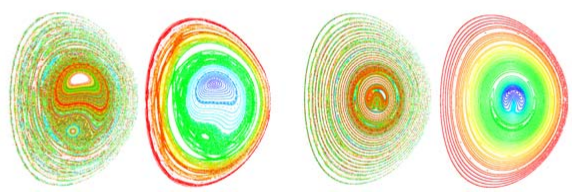


- Example of a recent 3D CEMM calculation using M3D code
- "Internal Kink" mode in a small tokamak (Sawtooth Oscillations)
- Good agreement between M3D, NIMROD, and experimental results
- 500 wallclock hours and over 200,000 CPU-hours

Excellent Agreement between NIMROD and M3D



Kinetic energy vs time in lowest toroidal harmonics



Flux Surfaces during crash at 2 times

M3D NIMROD M3D NIMROD

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Consider a simple 1-D Hyperbolic System of Equations (Wave Equation)

$$\left.\begin{aligned}\frac{\partial u}{\partial t} &= c\frac{\partial v}{\partial x}\\ \frac{\partial v}{\partial t} &= c\frac{\partial u}{\partial x}\end{aligned}\right\} \quad \frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0$$

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Implicit Centered Difference:

$$\frac{u_j^{n+1} - u_j^n}{\delta t} = c\left[\theta\left(\frac{v_{j+1/2}^{n+1} - v_{j-1/2}^{n+1}}{\delta x}\right) + (1-\theta)\left(\frac{v_{j+1/2}^{n} - v_{j-1/2}^{n}}{\delta x}\right)\right]$$

$$\frac{v_{j+1/2}^{n+1} - v_{j+1/2}^n}{\delta t} = c\left[\theta\left(\frac{u_{j+1}^{n+1} - u_j^{n+1}}{\delta x}\right) + (1-\theta)\left(\frac{u_{j+1}^{n} - u_j^{n}}{\delta x}\right)\right]$$

Stable for $\theta \geq ½$ 2nd order for $\theta = ½$

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Substitute from second equation into first:

$$u_j^{n+1}=u_j^n+(\delta tc)^2\left[\theta^2\left(\frac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{\delta x^2}\right)+\theta(1-\theta)\left(\frac{u_{j+1}^{n}-2u_j^{n}+u_{j-1}^{n}}{\delta x^2}\right)\right]+\delta tc\left(\frac{v_{j+1/2}^{n}-v_{j-1/2}^{n}}{\delta x}\right)$$

$$v_{j+1/2}^{n+1}=v_{j+1/2}^{n}+\frac{\delta tc}{\delta x}\left[\theta\left(u_{j+1}^{n+1}-u_j^{n+1}\right)+(1-\theta)\left(u_{j+1}^{n}-u_j^{n}\right)\right]$$

These two equations can be solved sequentially

Only first involves Matrix Inversion … Diagonally Dominant

$$u_j^{n+1}=u_j^n+(\delta tc)^2\left[\theta^2\left(\frac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{\delta x^2}\right)+\theta(1-\theta)\left(\frac{u_{j+1}^{n}-2u_j^{n}+u_{j-1}^{n}}{\delta x^2}\right)\right]+\delta tc\left(\frac{v_{j+1/2}^{n}-v_{j-1/2}^{n}}{\delta x}\right)$$

$$v_{j+1/2}^{n+1}=v_{j+1/2}^{n}+\frac{\delta tc}{\delta x}\left[\theta\left(u_{j+1}^{n+1}-u_j^{n+1}\right)+(1-\theta)\left(u_{j+1}^{n}-u_j^{n}\right)\right]$$

Rewrite using standard finite-difference notation:

$$\delta_x^2 u_j^{n+1}\equiv u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}$$
$$\delta_x v_j^n\equiv v_{j+1/2}^n-v_{j-1/2}^n$$
$$S\equiv\frac{\delta tc}{\delta x}$$

Operator to invert

$$\left[1-S^2\theta^2\delta_x^2\right]u_j^{n+1}=\left[1+S^2\theta(1-\theta)\delta_x^2\right]u_j^n+S\delta_x v_j^n$$

$$v_{j+1/2}^{n+1}=v_{j+1/2}^{n}+S\left[\theta\delta_x u_{j+1/2}^{n+1}+(1-\theta)\delta_x u_{j+1/2}^{n}\right]$$

This derivation can also be done in block matrix form:

Original equation:

$$\begin{bmatrix} \mathbf{I} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}^{n+1} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}^{n}$$

$$\mathbf{A}_{12} = \begin{bmatrix} x & & & \\ x & x & & \\ & x & x & \\ & & x & x \end{bmatrix} \quad \mathbf{A}_{21} = \begin{bmatrix} x & x & & \\ & x & x & \\ & & x & x \\ & & & x \end{bmatrix}$$

Lower and upper bi-diagonal matrices

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Lower and upper bi-diagonal matrices

Shur complement:

$$\mathbf{v}^{n+1} = -\mathbf{A}_{21}\mathbf{u}^{n+1} + \mathbf{B}_{21}\mathbf{u}^{n} + \mathbf{B}_{22}\mathbf{v}^{n}$$
$$\mathbf{u}^{n+1} = \mathbf{A}_{12}\mathbf{A}_{21}\mathbf{u}^{n+1} + \left(\mathbf{B}_{11} - \mathbf{A}_{12}\mathbf{B}_{21}\right)\mathbf{u}^{n} + \left(\mathbf{B}_{12} - \mathbf{A}_{12}\mathbf{B}_{22}\right)\mathbf{v}^{n}$$

or:

$$\boxed{\mathbf{I} - \mathbf{A}_{12}\mathbf{A}_{21}}\,\mathbf{u}^{n+1} = \left(\mathbf{B}_{11} - \mathbf{A}_{12}\mathbf{B}_{21}\right)\mathbf{u}^{n} + \left(\mathbf{B}_{12} - \mathbf{A}_{12}\mathbf{B}_{22}\right)\mathbf{v}^{n}$$
$$\mathbf{v}^{n+1} = -\mathbf{A}_{21}\mathbf{u}^{n+1} + \mathbf{B}_{21}\mathbf{u}^{n} + \mathbf{B}_{22}\mathbf{v}^{n}$$

Operator to invert (tridiagonal for 1D)

An alternate derivation:

$$\frac{\partial u}{\partial t} = c\frac{\partial v}{\partial x}$$
$$\frac{\partial v}{\partial t} = c\frac{\partial u}{\partial x}$$

Expand RHS in Taylor series in time to time-center

$$\frac{\partial u}{\partial t} = c\frac{\partial}{\partial x}\left[v^n + \theta\delta t\frac{\partial v}{\partial t}\right]$$
$$\frac{\partial v}{\partial t} = c\frac{\partial}{\partial x}\left[u^n + \theta\delta t\frac{\partial u}{\partial t}\right]$$

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Substitute from second equation into first

$$\frac{\partial u}{\partial t} = c\frac{\partial}{\partial x}\left[v^n + \theta\delta t\left(c\frac{\partial}{\partial x}\left[u^n + \theta\delta t\frac{\partial u}{\partial t}\right]\right)\right]$$
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$$\frac{\partial v}{\partial t} = c\frac{\partial}{\partial x}\left[u^n + \theta\delta t\frac{\partial u}{\partial t}\right]$$

Use standard centered difference in time:

$$\left[1-\theta^2(\delta t)^2c^2\frac{\partial^2}{\partial x^2}\right]u^{n+1} = \left[1+\theta(1-\theta)(\delta t)^2c^2\frac{\partial^2}{\partial x^2}\right]u^n + \delta tc\frac{\partial}{\partial x}v^n$$
$$v^{n+1} = v^n + \delta tc\left[\theta\frac{\partial}{\partial x}u^{n+1} + (1-\theta)\frac{\partial}{\partial x}u^n\right]$$

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$$v^{n+1} = v^n + \delta tc\left[\theta\frac{\partial}{\partial x}u^{n+1} + (1-\theta)\frac{\partial}{\partial x}u^n\right]$$

This is the same operator as before when centered spatial differences used

Now apply this technique to the basic 3D MHD equations:

$$\rho_0 \dot{\mathbf{V}} = \frac{1}{\mu_0}\left[\nabla \times \mathbf{B}\right] \times \mathbf{B} - \nabla p$$
$$\dot{\mathbf{B}} = \nabla \times \left[\mathbf{V} \times \mathbf{B}\right]$$
$$\dot{p} = -\mathbf{V} \cdot \nabla p - \gamma p \nabla \cdot \mathbf{V}$$

Ideal MHD Equations for velocity, magnetic field, and pressure:

Symmetric Hyperbolic System

7-waves

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$$\rho_0 \dot{\mathbf{V}} = \frac{1}{\mu_0}\left[\nabla \times \left(\mathbf{B} + \theta \delta t \dot{\mathbf{B}}\right)\right] \times \left(\mathbf{B} + \theta \delta t \dot{\mathbf{B}}\right) - \nabla\left(p + \theta \delta t \dot{p}\right)$$
$$\dot{\mathbf{B}} = \nabla \times \left[\left(\mathbf{V} + \theta \delta t \dot{\mathbf{V}}\right) \times \mathbf{B}\right]$$
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Taylor Expand in Time as before

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Taylor Expand in Time as before

Substitute from 2nd and 3rd equation into first, finite difference in time:

$$\left\{\rho - \theta^2 (\delta t)^2 L\right\} \mathbf{V}^{n+1} = \left\{\rho - \theta(\theta-1)(\delta t)^2 L\right\} \mathbf{V}^n + \delta t \left\{-\nabla p + \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B}\right\}$$

MHD Operator: →

$$L\{\mathbf{V}\} = \frac{1}{\mu_0}\left\{\nabla \times \left[\nabla \times \left(\mathbf{V} \times \mathbf{B}\right)\right]\right\} \times \mathbf{B} + \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \left[\nabla \times \left(\mathbf{V} \times \mathbf{B}\right)\right] + \nabla\left(\mathbf{V} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{V}\right)$$

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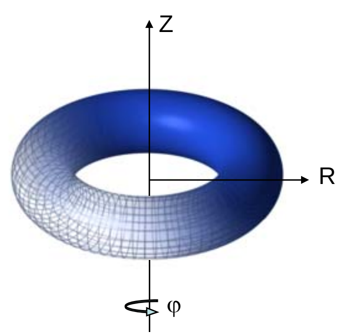
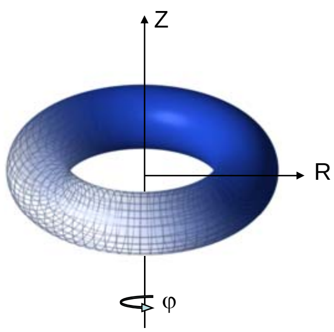
$$L\{\mathbf{V}\} = \frac{1}{\mu_0}\left\{\nabla \times \left[\nabla \times \left(\mathbf{V} \times \mathbf{B}\right)\right]\right\} \times \mathbf{B} + \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \left[\nabla \times \left(\mathbf{V} \times \mathbf{B}\right)\right] + \nabla\left(\mathbf{V} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{V}\right)$$



- Need to solve this in 3D torus with strong magnetic field in toroidal direction (φ) … anisotropy
- Wide range of wave speeds leads to ill-conditioned matrices
- Gradients in (R,Z) plane much larger than in φ direction
- Also need to preserve $\nabla \cdot \mathbf{B} = 0$

$$\left\{\rho-\theta^2(\delta t)^2 L\right\}\mathbf{V}^{n+1}=\left\{\rho-\theta(\theta-1)(\delta t)^2 L\right\}\mathbf{V}^n+\delta t\left\{-\nabla p+\frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B}\right\}$$

$$L\{\mathbf{V}\}=\frac{1}{\mu_0}\left\{\nabla\times\left[\nabla\times(\mathbf{V}\times\mathbf{B})\right]\right\}\times\mathbf{B}+\frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\left[\nabla\times(\mathbf{V}\times\mathbf{B})\right]$$
$$+\nabla\left(\mathbf{V}\cdot\nabla p+\gamma p\nabla\cdot\mathbf{V}\right)$$



CEMM has 3 Major Codes using different approaches for solving this equation:

NIMROD: Fourier analyses in φ and solves each harmonic separately using SuperLU with GMRES to couple harmonics

M3D: Finite Differences in φ, but only solves implicit operator equation in (R,Z) planes, using either GMRES or HYPRE (via PetSC) Explicit differences in φ.

M3D-C^1: Finite Differences in φ, uses SuperLU in (R,Z) plane, now going to 3D.

M3D-C^1 uses a stream function / potential form of the velocity and magnetic fields

$$\vec{V}=\nabla U\times\hat{z}+\nabla_\perp\chi+V_z$$

$$\vec{B}=\nabla\psi\times\hat{z}+I\hat{z}$$

The sparse matrix equation to be solved for the velocity variables take the form:

$$\begin{bmatrix} S_{11}^v & S_{12}^v & S_{13}^v \\ S_{21}^v & S_{22}^v & S_{23}^v \\ S_{31}^v & S_{32}^v & S_{33}^v \end{bmatrix}\cdot\begin{bmatrix} U \\ V_z \\ \chi \end{bmatrix}^{n+1}=\begin{bmatrix} D_{11}^v & D_{12}^v & D_{13}^v \\ D_{21}^v & D_{22}^v & D_{23}^v \\ D_{31}^v & D_{32}^v & D_{33}^v \end{bmatrix}\cdot\begin{bmatrix} U \\ V_z \\ \chi \end{bmatrix}^{n}+\begin{bmatrix} R_{11}^v & R_{12}^v & R_{13}^v \\ R_{21}^v & R_{22}^v & R_{23}^v \\ R_{31}^v & R_{32}^v & R_{33}^v \end{bmatrix}\cdot\begin{bmatrix} \psi \\ I \\ P_e \end{bmatrix}^{n}$$

- Corresponds to the operator equation derived on earlier vg:

$$\left\{\rho-\theta^2(\delta t)^2 L\right\}\mathbf{V}^{n+1}=\left\{\rho-\theta(\theta-1)(\delta t)^2 L\right\}\mathbf{V}^n+\delta t\left\{-\nabla p+\frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B}\right\}$$

- Also contains 2 non-trivial sub-systems (reduced MHD)

$$\left[S_{11}^v\right]\cdot\left[U\right]^{n+1}=\left[D_{11}^v\right]\cdot\left[U\right]^{n}+\left[R_{11}^v\right]\cdot\left[\psi\right]^{n}$$ etc.

In 2D, Implicit equations are solved using SuperLU_Dist

Details for bassi.nersc.gov:

Mesh points	180 x 180
Matrix Rank	5.9 x 10^5
# Non-zeros	9.5 x 10^7
# NZ in L/U	8.8 x 10^8

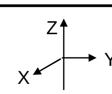
# processors	8	32	128
Factor (s)	69.5	38.1	16.9
Gflop/s	27.2	50.1	112.8*

Total problem time (8 processors) for typical high resolution reconnection problem = 208 s x 400 cycles x 8p = 185 p-hrs

Note that for linear problem, Matrix need only be factored once. For semi-implicit method, matrix needs to be factored only occasionally.

*NOTE: In 3D, if we had 100 planes with simultaneous instances of SuperLU, this would be 12,800 p and 11.2 Tflop/s actual!

How to take this to 3D ?



note: "Poisson Bracket"

$$[a,b] \equiv \frac{\partial a}{\partial X}\frac{\partial b}{\partial Y} - \frac{\partial a}{\partial X}\frac{\partial b}{\partial Y}$$

"Inner Product"

$$(a,b) \equiv \frac{\partial a}{\partial X}\frac{\partial b}{\partial X} + \frac{\partial a}{\partial Y}\frac{\partial b}{\partial Y}$$

Consider (1,1) component in 3D: Corresponds to "Strauss Equations" (3D reduced MHD for ψ and U)

$$\nabla_\perp^2 \dot{U} + \left[\nabla_\perp^2 U, U\right] = \left[\nabla_\perp^2 \psi, \psi\right] + B\frac{\partial}{\partial z}\nabla_\perp^2 \psi$$

$$\dot{\psi} + \left[\psi, U\right] = B\frac{\partial}{\partial z}U$$

Taylor expand in time:

$$\nabla_\perp^2 \dot{U} + \left[\nabla_\perp^2 U + \theta\delta t \nabla_\perp^2 \dot{U}, U + \theta\delta t \dot{U}\right] = \left[\nabla_\perp^2 \psi + \theta\delta t \nabla_\perp^2 \dot{\psi}, \psi + \theta\delta t \dot{\psi}\right] + B\frac{\partial}{\partial z}\nabla_\perp^2 \psi + \theta\delta t B\frac{\partial}{\partial z}\nabla_\perp^2 \dot{\psi}$$

$$\dot{\psi} + \left[\psi, U + \theta\delta t \dot{U}\right] = B\frac{\partial}{\partial z}U + \theta\delta t B\frac{\partial}{\partial z}\dot{U}$$

Substitute field derivatives into velocity equation to get implicit equation:

Finite Difference in Z: $\dot{U}_z = \frac{1}{\delta z}\left[\dot{U}_{j+1} - \dot{U}_{j-1}\right]$ $\dot{U}_{zz} = \frac{1}{(\delta z)^2}\left[\dot{U}_{j+1} - 2\dot{U}_j + \dot{U}_{j-1}\right]$

When we apply C^1 continuous finite elements in *(x,y)*, we get a block tridiagonal equation, with the matrix blocks being 2D matrices:

$$B_j^0 U_j^{n+1} + D_j^0 + \varepsilon(A_j^1 U_{j+1}^{n+1} + B_j^1 U_j^{n+1} + C_j^1 U_{j-1}^{n+1} + D_j^1) = 0$$

$$B_j^0 U_j^{n+1} = \nu_i \nabla_\perp^2 \dot{U}_j - (\theta\delta t)^2 \left[+\nabla_\perp^2 \psi \left[\nu_i, \left[\psi, U_j^{n+1}\right]\right] - \left(\left[\psi, U_j^{n+1}\right], \left[\nu_i, \psi\right]\right) + B\left(\nu_i, \left[\psi_z, U_j^{n+1}\right]\right)\right]$$

$$B_j^1 U_j^{n+1} = \nu_i \nabla_\perp^2 \dot{U}_j - (\theta\delta t)^2 \left[-\frac{2B^2}{(\delta z)^2} \nu_i \nabla_\perp^2 U_j^{n+1}\right]$$

$$A_j U_{j+1}^{n+1} = -(\theta\delta t)^2 \left[\begin{array}{l} -\frac{B}{\delta z}\nabla_\perp^2 \psi\left[\nu_i, U_{j+1}^{n+1}\right] - \frac{B}{\delta z}\nabla_\perp^2 U_{j+1}^{n+1}\left[\nu_i, \psi\right] \\ +\frac{B^2}{(\delta z)^2}\nu_i \nabla_\perp^2 U_{j+1}^{n+1} + \frac{B}{\delta z}\left(\nu_i, \left[\psi, U_{j+1}^{n+1}\right]\right)\end{array}\right]$$

$$C_j U_{j-1}^{n+1} = -(\theta\delta t)^2 \left[\begin{array}{l} \frac{B}{\delta z}\nabla_\perp^2 \psi\left[\nu_i, U_{j-1}^{n+1}\right] + \frac{B}{\delta z}\nabla_\perp^2 U_{j-1}^{n+1}\left[\nu_i, \psi\right] \\ +\frac{B^2}{(\delta z)^2}\nu_i \nabla_\perp^2 U_{j-1}^{n+1} - \frac{B}{\delta z}\left(\nu_i, \left[\psi, U_{j-1}^{n+1}\right]\right)\end{array}\right]$$

Can this structure be used to define an efficient iteration scheme where the 2D direct solves serve as a preconditioner ?

Summary of Proposed 3D Time Advance

$$B_j^0 U_j^{n+1} + D_j^0 + \varepsilon(A_j^1 U_{j+1}^{n+1} + B_j^1 U_j^{n+1} + C_j^1 U_{j-1}^{n+1} + D_j^1) = 0$$

U_j^{n+1} is vector of all unknown velocities on plane j at new time

$B_j^0, A_j^1, B_j^1, C_j^1$ are 2D sparse matrices at plane j

D_j^0, D_j^1 are 2D vectors at plane j

Possible iteration scheme. Use SuperLU to factor the B_j^0 simultaneously

$$U_j^{i+1} = -\left[B_j^0\right]^{-1} \left[D_j^0 + \varepsilon(A_j^1 U_{j+1}^i + B_j^1 U_j^i + C_j^1 U_{j-1}^i + D_j^1)\right]$$

Note that B_j^0 matrices only need to be factored once per timestep

Other Efforts in CEMM

NIMROD:

Sovinec (U. Wisc):

- Now using "poloidal preconditioning". Exploring 'toroidal preconditioning

S. Vadlamani, S. Kruger (Tech X), T. Manteuffel,S. McCormick (CU APPM):

- SBIR contract to explore the use of MultiGrid on the MHD equations. Plan to use HYPRE via the PETSc interface